

# Lecture 07: Graph Representation

# Objective

- We shall develop a new graph representation to argue security and correctness of cryptographic schemes
- As a representative application of this notation, we shall analyze private-key Encryption schemes using graphs

# Assumption about Private-key Encryption Schemes

For simplicity of proof and clarity of the intuition, we shall consider the class of all private-key encryption algorithms with the following restrictions

- 1 The key-generation algorithm  $\text{Gen}$  outputs a secret key sampled uniformly at random from the set  $\mathcal{K}$
- 2 The encryption algorithm  $\text{Enc}_{\text{sk}}(m)$  is deterministic

I want to emphasize that with a bit of effort these *restrictions* can be removed

# Graph of Private-key Encryption

Suppose  $(\text{Gen}, \text{Enc}, \text{Dec})$  is a private-key encryption scheme that satisfies the two restrictions we mentioned earlier. We construct the following bipartite graph

- The left partite set is the set of all message  $\mathcal{M}$
- The right partite set is the set of all cipher-texts  $\mathcal{C}$
- Given a message  $m \in \mathcal{M}$  and a cipher-text  $c \in \mathcal{C}$ , we add an edge  $(m, c)$  labeled  $sk$ , if we have  $c = \text{Enc}_{sk}(m)$

This is the graph corresponding to the encryption scheme  $(\text{Gen}, \text{Enc}, \text{Dec})$

**Intuition.** The edge labeled  $sk$  witnesses the fact that the message  $m$  is encrypted to the cipher-text  $c$ . Or, we write this as  $m \xrightarrow{sk} c$ . We emphasize that there might be more than one secret key that witnesses the fact that the message  $m$  is encrypted to the cipher-text  $c$ . Let  $\text{wt}(m, c)$  represent the number of secret keys  $sk$  such that  $sk$  witnesses the fact that  $c$  is an encryption of  $m$

# Describing Private-key Encryption Schemes

- Till now we have represented private-key encryption scheme as a triplet of algorithms (Gen, Enc, Dec)
- Henceforth, we can equivalently express them as graphs

# Property One: Characterization of Correctness

## Theorem

A private-key encryption scheme  $(\text{Gen}, \text{Enc}, \text{Dec})$  is incorrect if and only if there are two distinct messages  $m, m' \in \mathcal{M}$ , a secret key  $\text{sk} \in \mathcal{K}$ , and a cipher-text  $c \in \mathcal{C}$  such that  $m \xrightarrow{\text{sk}} c$  and  $m' \xrightarrow{\text{sk}} c$ .

- Note that if there are two message  $m, m'$  such that  $m \xrightarrow{\text{sk}} c$  and  $m' \xrightarrow{\text{sk}} c$  then Bob cannot distinguish whether Alice produced the cipher text  $c$  for the message  $m$  or  $m'$ . Hence, whatever decoding Bob performs, he is bound to be incorrect
- For the other direction, suppose Bob is unable to decode the  $(\text{sk}, c)$  correctly. If there is a unique  $m \in \mathcal{M}$  such that  $m \xrightarrow{\text{sk}} c$  then Bob can obviously decode correctly. So, there must be two different messages  $m, m' \in \mathcal{K}$  such that  $m \xrightarrow{\text{sk}} c$  and  $m' \xrightarrow{\text{sk}} c$

## Property Two: Correct Schemes Cannot Compress I

### Theorem

A correct private-key encryption scheme  $(\text{Gen}, \text{Enc}, \text{Dec})$  has  $|\mathcal{C}| \geq |\mathcal{M}|$ .

- Suppose not. That is, assume that we have a correct private-key encryption scheme with  $|\mathcal{C}| < |\mathcal{M}|$ .
- Fix any secret key  $\text{sk} \in \mathcal{K}$ .
- Suppose  $\mathcal{M} = \{m_1, m_2, \dots, m_\alpha\}$ . Consider the following maps

$$m_1 \xrightarrow{\text{sk}} c_1$$

$$m_2 \xrightarrow{\text{sk}} c_2$$

$$\vdots$$

$$m_\alpha \xrightarrow{\text{sk}} c_\alpha$$

## Property Two: Correct Schemes Cannot Compress II

Note that these mappings exist because given any  $sk$  and  $m$  the encryption algorithm maps to a unique cipher-text.

- Since  $|\mathcal{C}| < |\mathcal{M}|$ , by pigeon-hole principle there are two distinct messages  $m, m' \in \mathcal{M}$  and a cipher text  $c \in \mathcal{C}$  such that  $m \xrightarrow{sk} c$  and  $m' \xrightarrow{sk} c$
- So the scheme is incorrect. Hence contradiction.



## Property Three: Characterization of Security I

### Theorem

*A private-key encryption scheme  $(\text{Gen}, \text{Enc}, \text{Dec})$  is secure if and only if for any  $c$  and two distinct message  $m, m' \in \mathcal{M}$  we have  $\text{wt}(m, c) = \text{wt}(m', c)$ .*

- For any  $m \in \mathcal{M}$  and  $c \in \mathcal{C}$ , note that we have  $\mathbb{P}[\mathbb{C} = c | \mathbb{M} = m] = \text{wt}(m, c) / |\mathcal{K}|$ .
- Exercise: Prove that the security definition we have studied is equivalent to saying the following  
“For any two distinct messages  $m, m' \in \mathcal{M}$  and a cipher-text  $c \in \mathcal{C}$  we have:  $\mathbb{P}[\mathbb{C} = c | \mathbb{M} = m] = \mathbb{P}[\mathbb{C} = c | \mathbb{M} = m']$ ”
- Given this result, we can conclude that a scheme  $(\text{Gen}, \text{Enc}, \text{Dec})$  is secure if and only if  
“For any two distinct messages  $m, m' \in \mathcal{M}$  and a cipher-text  $c \in \mathcal{C}$  we have:  $\text{wt}(m, c) = \text{wt}(m', c)$ ”

## Property Three: Characterization of Security II

- **Food for thought.** In a secure scheme, if there are  $m \xrightarrow{\text{sk}} c$ , then for all  $m' \in \mathcal{M}$  there exists some  $\text{sk}'$  such that  $m' \xrightarrow{\text{sk}'} c$
- **Food for thought.** The size of the set  $\mathcal{K}$  need not be divisible by the size of the set  $\mathcal{M}$ . However, if there is a message  $m$  and a cipher-text  $c$  such that  $\text{wt}(m, c) = w$ , then the number of secret keys  $|\mathcal{K}| \geq w|\mathcal{M}|$ . Why?

## Theorem

A correct and secure private-key encryption scheme (Gen, Enc, Dec) has  $|\mathcal{K}| \geq |\mathcal{M}|$

- Suppose not. That is, there is a correct and secure scheme with  $|\mathcal{K}| < |\mathcal{M}|$ .
- Fix a cipher-text  $c \in \mathcal{C}$  such that there exists  $m \in \mathcal{M}$  and  $sk \in \mathcal{K}$  such that  $m \xrightarrow{sk} c$ . Intuitively, we are picking a cipher-text that has a positive probability. For example, we are not picking a cipher-text that is never actually produced.
- Let the message space be  $\mathcal{M} = \{m_1, m_2, \dots, m_\alpha\}$
- Note that, for any  $m_i \in \mathcal{M}$  there exists some  $sk_i$  such that  $m_i \xrightarrow{sk_i} c$  (This is a property of secure private-key encryption schemes that was left as an exercise in the previous slide)

# Property Four: Correct+Secure Schemes need Lots of Keys II

- Now, consider the mappings

$$m_1 \xrightarrow{\text{sk}_1} c$$

$$m_2 \xrightarrow{\text{sk}_2} c$$

⋮

$$m_\alpha \xrightarrow{\text{sk}_\alpha} c$$

- Since  $|\mathcal{K}| < |\mathcal{M}|$ , by pigeon-hole principle, there exists two distinct messages  $m_i, m_j$  such that  $\text{sk}_i = \text{sk}_j$  in the above mappings.
- This violates correctness. Hence contradiction.

# Optimality of One-time Pad

- Note that any correct private-key encryption scheme must have  $|\mathcal{C}| \geq |\mathcal{M}|$  (property two)
- Note that any correct and secure private-key encryption scheme must have  $|\mathcal{K}| \geq |\mathcal{M}|$  (property four)
- One-time pad is a correct and secure scheme that achieves  $|\mathcal{K}| = |\mathcal{C}| = |\mathcal{M}|$

## Additional Food for Thought

- Recall that Property four states that the “correctness and security” of a private-key encryption scheme implies that the size of the set of keys is greater-than-or-equal to the size of the set of messages. For any  $\mathcal{M}$ , construct a correct but insecure private-key encryption scheme such that  $|\mathcal{K}| = 1$ ! This result shall show the necessity of both correctness and security in that property.
- Another natural question is: Can we provide such guarantees for private-key encryption schemes that are secure but incorrect? The answer is NO. Think of a private-key encryption scheme that is secure (but incorrect) and works for any message set  $\mathcal{M}$  and has  $|\mathcal{K}| = |\mathcal{C}| = 1$ !